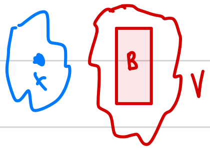
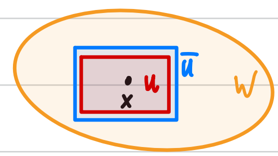
Math 451: Introduction to General Topology
Lecture 17

○ T_3 (= regular) if it is T_1 and for each $x \in X$ and closed set $B \subseteq X$ with $x \notin B$ $\exists$ disjoint open $U \ni x$ and $V \supseteq B$.

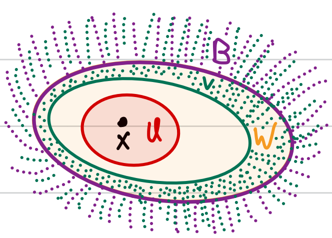


Note. Because singletons are closed in T_1, $T_3 \Rightarrow T_2$.

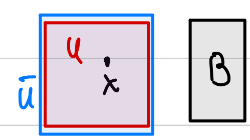
Rephrasing of regularity. A T_1 top. space X is regular $\Leftrightarrow$ $\forall x \in X$ and $\forall$ open $W \ni x$ $\exists$ open U s.t. $x \in U \subseteq \overline{U} \subseteq W$.



Proof. $\Rightarrow$. Given $x \in X$ and open $W \ni x$, then $B := W^c$ is closed and $x \notin B$, so regularity gives disjoint open sets $U \ni x$ and $V \supseteq B = W^c$, hence $x \in U \subseteq V^c \subseteq W$. Because V^c is closed, $\overline{U} \subseteq V^c$ so $x \in U \subseteq \overline{U} \subseteq V^c \subseteq W$.



$\Leftarrow$. Fix $x \in X$ and a closed set $B \not\ni x$. Then $W := B^c$ is an open neighbourhood of x, so $\exists$ open U s.t. $x \in U \subseteq \overline{U} \subseteq W$. Then $V := \overline{U}^c$ is open, $V \supseteq W^c = B$, and V is disjoint from U. □



Although this is just a rephrasing on the level of definitions, it helps thinking about regularity in terms of open neighbourhoods.

Cor. Every 0-dimensional T_1 top. space is regular.

Proof. Let $\mathcal{B}$ be a basis of clopen sets. Then for each $x \in X$ and open $W \ni x$, there is a clopen $U \in \mathcal{B}$ s.t. $x \in U \subseteq W$ since W is a union of sets in $\mathcal{B}$. But then $x \in U \subseteq \overline{U} \subseteq W$ because $U = \overline{U}$ since U is closed. □

Examples. (a) For any nonempty Σ, $\Sigma^{\mathbb{N}}$ is 0-dim and Hausdorff, so regular. In fact, this is a metric space and all metric spaces are regular (in fact, normal).

(b) Sorgenfrey line is 0-dim since the sets $[a,b)$ form a basis for the top and these are clopen sets since $[a,b)^c = (-\infty, a) \cup [b, \infty)$ which is open since it is a union of open sets. Hence, Sorgenfrey line is regular.

Example of T_2 but not T_3. Let τ be the usual (Euclidean) top on $\mathbb{R}$. Take

$$B := \{\tfrac{1}{n} : n \in \mathbb{N}^+\}.$$

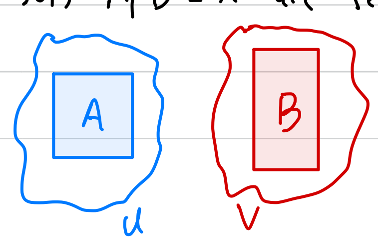
We make this set closed by adding B^c to the topology, i.e. let τ' be the topology generated by B^c together with τ. This richer top τ' is still Hausdorff (T_2) because τ was already Hausdorff.

Claim. τ' is not regular.

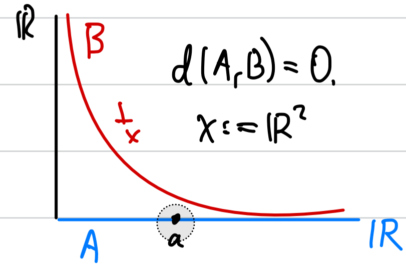
Proof. Let $x := 0$ and B as above. B is closed in τ' and $x \notin B$, so we only need to show that 0 and $W := B^c$ fail the rephrasing of regularity. Indeed any open U with $0 \in U \subseteq W$ contains a set $(a,b) \cap W$, where $a < 0 < b$. The closure of $(a,b) \cap W$ is $[a,b]$ because $(a,b) \cap W = (a,b) \setminus B$ and each $\frac{1}{n}$ in (a,b) adheres to (a,b). So $\overline{U} \supseteq [a,b]$ but $[a,b] \ni \frac{1}{n}$ for some large enough $n \in \mathbb{N}^+$, so $\overline{U} \cap B \neq \emptyset$, i.e. $\overline{U} \not\subseteq W$. □

Remark. Because T_0, T_1, T_2 are only about separating points by open sets, they are upward hereditary, i.e. if τ is a T_i topology on X, $i \leq 2$, and $\tau' \supseteq \tau$ is a finer topology then τ' is also T_i. However, this is not true for T_3, as the above example shows, because although τ' has more open sets (so easier to separate things), it also has more closed sets (so more things to separate).

- T_4 (= normal) if it is T_1 and any two disjoint closed sets $A, B \subseteq X$ are separated by open sets, i.e. $\exists$ disjoint open $U \supseteq A$ and $V \supseteq B$.



Note. T_4 implies T_3 because each point is a closed set due to T_1.



Prop. Metrizable spaces are normal.

Proof. Let X be a top space with a metric d generating the topology. Let $A, B \subseteq X$ be disjoint closed sets. For each $a \in A$, since $a \notin B$ and B is closed, $d(a,B) := \inf_{b \in B} d(a,b) > 0$.
Let $U := \bigcup_{a \in A} B_{r_a}(a)$, where $r_a := \frac{1}{2} d(a,B)$. Similarly, for each $b \in B$, $d(A,b) > 0$ and we let $V := \bigcup_{b \in B} B_{r_b}(b)$, where $r_b := \frac{1}{2} d(A,b)$. Clearly $U \supseteq A$ and $V \supseteq B$ are open sets. It remains to verify that $U \cap V = \emptyset$, but if $U \cap V \neq \emptyset$, then $\exists\, a \in A$, $b \in B$ s.t. $B_{r_a}(a) \cap B_b(b) \neq \emptyset$, so $d(a,b) < r_a + r_b = \frac{1}{2} d(a,B) + \frac{1}{2} d(A,b) \leq \frac{1}{2} d(a,b) + \frac{1}{2} d(a,b) = d(a,b)$, a contradiction. $\square$

Prop. Regular Lindelöf top spaces are normal.

Proof. Outlined in HW.

Example. Sorgenfrey line $\mathbb{R}$ is normal because it is Lindelöf and regular.

Example of T_3 but not T_4. Consider the Sorgenfrey plane $\mathbb{R}^2$ with the product topology of Sorgenfrey topologies on $\mathbb{R}$, equivalently, the top on $\mathbb{R}^2$ generated by sets $[a,b) \times [c,d)$, which form a basis because they are closed under finite intersections. Product of two regular spaces is regular, so Sorgenfrey plane is regular. This can also be seen directly from the fact that $[a,b) \times [c,d)$ are clopen, hence Sorgenfrey plane is also 0-dim, hence regular.
However, it is not normal. This is a bit involved and will be outlined in HW.

Continuity.

Def. Let X, Y be top. spaces. A function $f : X \to Y$ is said to be

- **continuous at $x_0 \in X$** if for every open $V \ni f(x_0)$ $\exists$ open $U \ni x_0$ with $U \subseteq f^{-1}(V)$; i.e. $f^{-1}(V)$ is a (not necessarily open) neighbourhood of x_0.
- **continuous** if it is continuous at every point $x_0 \in X$; equivalently (by the same proof as for metric spaces), f-preimages of open sets are open.

Prop. Let X, Y be top. spaces. A function $f: X \to Y$ is continuous $\iff$ f-preimages of a generating collection of open sets are open, i.e. for some prebasis $\mathcal{S}$ for the top. of Y, $f^{-1}(V)$ is open for each $V \in \mathcal{S}$.

Proof. This is simply because preimages respect/commute with intersections and unions. In detail, let V be an open set, so it is an (arbitrary) union of finite intersections of sets in $\mathcal{S}$. Then $f^{-1}(V)$ is an (arbitrary) union of finite intersections of f-preimages of sets in $\mathcal{S}$, hence are open. $\square$